

# TIGHT CONTACT STRUCTURES ON HYPERBOLIC THREE-MANIFOLDS

M. FIRAT ARIKAN AND MERVE SEÇGİN

**ABSTRACT.** We show the existence of tight contact structures on infinitely many hyperbolic three-manifolds obtained via Dehn surgeries along sections of hyperbolic surface bundles over circle.

## 1. INTRODUCTION

A *contact three-manifold* is a pair  $(M, \xi)$  where  $M$  is a smooth 3-manifold and  $\xi \subset TM$  is a totally non-integrable 2-plane field distribution on  $M$ . Here we always assume that  $\xi$  is a *co-oriented positive* contact structure, that is,  $\xi = \text{Ker}(\alpha)$  for a *contact* 1-form  $\alpha$  satisfying  $\alpha \wedge d\alpha > 0$  with respect to a pre-given orientation on  $M$ . A disk  $D$  in a contact 3-manifold  $(M, \xi)$  is called *overtwisted* if the boundary circle  $\partial D$  is tangent to  $\xi$  everywhere. A contact structure  $\xi$  is called overtwisted if there is an *overtwisted* disk in  $(M, \xi)$ , otherwise it is called *tight*. It is known that every closed oriented 3-manifold admits an overtwisted contact structure ([7], [19]). On the other hand, there are 3-manifolds which do not admit a tight contact structure [10].

There are some classification results on tight contact structures with respect to the geometric speciality of 3-manifolds. Lisca and Stipsicz in [18] proved that a closed oriented Seifert fibered 3-manifold admits a tight contact structure if and only if it is not gotten  $(2q-1)$ -surgery along the  $(2, 2q+1)$  torus knot in  $S^3$  for  $q \geq 1$ . In two independent work ([2], [16]), they showed the existence of tight contact structures on toroidal 3-manifolds. It is known that every irreducible 3-manifold that is neither toroidal nor Seifert fibered is hyperbolic. Kaloti and Tosun in [17] find infinitely many hyperbolic rational homology spheres admitting tight contact structures. Eteü in [9] also explored that infinitely many hyperbolic 3-manifolds that carry tight contact structures. His construction uses Dehn surgeries along sections of hyperbolic torus bundles over  $S^1$ . Here we'll follow similar ideas for surface bundles over  $S^1$  with fiber genus at least two.

Let  $\Sigma_g$  be a closed connected orientable surface with genus  $g$ . In this paper assume that  $g$  is always greater than 1. We will denote  $MCG(\Sigma_g)$  by the *mapping class group* of  $\Sigma_g$ , i.e, the group of isotopy classes of orientation preserving homeomorphisms of  $\Sigma_g$ . Let  $t_a$  be the positive Dehn twist along a simple closed curve  $a$ .

Let  $\phi \in MCG(\Sigma_g)$  be the mapping class representing the homeomorphism

$$(1) \quad t_{a_1}^m t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n$$

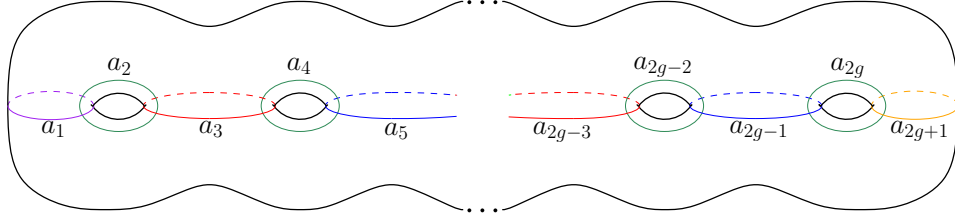
where  $a_i$ 's are simple closed curves on  $\Sigma_g$  as indicated in Figure 1.

---

*Date:* December 9, 2016.

*2000 Mathematics Subject Classification.* 57R65, 58A05, 58D27.

*Key words and phrases.* contact structure, tight, Stein fillable, open book, hyperbolic manifold.

FIGURE 1. Simple closed curves on the surface  $\Sigma_g$ .

Denote by  $M_\phi$  the *mapping torus* with fibers  $\Sigma_g$  and monodromy  $\phi$ . Let  $M_\phi(r)$  be the surgered manifold obtained by performing rational  $r$ -surgery along a section of  $M_\phi$ . Clearly,  $\phi$  has a fixed point, so such a section exists. The following theorems give examples required:

**Theorem 1.1.** *Suppose  $g \geq 2$ ,  $m, n \in \mathbb{Z}$ ,  $r \in \mathbb{Q}$  and  $\phi$  as indicated in (1). Then  $M_\phi(r)$  is hyperbolic for all but finitely many  $m$  and  $r$ .*

**Theorem 1.2.** *Suppose  $g \geq 1$ ,  $r \in \mathbb{Q}$  and  $\phi$  as indicated in (1). Then  $M_\phi(r)$  admits a tight contact structure  $\xi$  for any  $m, n \in \mathbb{Z}^+$  and for all  $r \neq 2g - 1$ .*

As a consequence of the theorems we have:

**Corollary 1.3.** *Suppose  $g \geq 2$ ,  $m, n \in \mathbb{Z}^+$ ,  $r \in \mathbb{Q}$  and  $\phi$  as indicated in (1). Then  $M_\phi(r)$  is a hyperbolic manifold admitting a tight contact structure for all  $r \neq 2g - 1$  and all but finitely many  $m \in \mathbb{Z}^+$ .  $\square$*

The proof of Theorem 1.1 and Theorem 1.2 will be given in Section 2 and Section 3.

## 2. PROOF OF THEOREM 1.1

In order to prove the theorem, we'll focus on pseudo-Anosov homeomorphisms and construct infinitely many hyperbolic 3-manifolds via pseudo-Anosov monodromies. A hyperbolic 3-manifold is a 3-manifold which admits a complete finite-volume hyperbolic structure. Thurston [22] demonstrated that an orientable surface bundle over circle whose fiber is a compact surface of negative Euler characteristic is hyperbolic if and only if the monodromy of the surface bundle is a pseudo-Anosov homeomorphism. Another deep result of Thurston is hyperbolic Dehn surgery theorem which states that a hyperbolic 3-manifold remains hyperbolic after Dehn filling along a link for all slopes except finitely many of them (For details see [23]). In order to apply these results, we need a lemma where we construct infinitely many pseudo-Anosov diffeomorphisms as products of certain Dehn twists:

**Lemma 2.1.** *Let  $\phi$  be the class in  $MCG(\Sigma_g)$  as described in (1) above. Then  $\phi$  is pseudo-Anosov for any integer  $n$  and for all but at most 7 consecutive values of  $m$ .*

Denote by  $\iota(\alpha, \beta)$  geometric intersection number of the curves  $\alpha$  and  $\beta$ . We say a set of simple closed curves  $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$  fills  $\Sigma_g$  if  $\Sigma_g \setminus \{\gamma_1, \gamma_2, \dots, \gamma_k\}$  is a disjoint union of topological disks. In order to prove Lemma 2.1, we use the following theorem of Fathi:

**Theorem 2.2.** ([12]) *Let  $f$  be the class in  $MCG(\Sigma_g)$  and let  $\gamma$  be a simple closed curve in  $\Sigma_g$ . If the orbit of  $\gamma$  under  $f$  fills  $\Sigma_g$ , then  $t_\gamma^m f$  is a pseudo-Anosov class except for at most 7 consecutive values of  $m$ .*

*Proof of Lemma 2.1.* Let  $\gamma$  represents the curve  $a_1$  and let  $f$  be the product of Dehn twists  $t_{a_1} t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n$ . Then conclude that

$$f(\gamma) = t_{a_1} t_{a_2}(a_1) = a_2, \quad f^2(\gamma) = t_{a_1} t_{a_2} t_{a_3}(a_2) = a_3,$$

and inductively,

$$f^i(\gamma) = a_{i+1} \text{ for all } i \in 1, 2, \dots, 2g - 1.$$

Since the complement  $\Sigma_g \setminus \{a_1, \dots, a_{2g}\}$  is a topological disk, we can say the orbit of  $\gamma$  under  $f$  fills  $\Sigma_g$ . As a result of Theorem 2.2,  $\phi$  is pseudo-Anosov except for at most 7 consecutive  $m$  values.  $\square$

Now we have a family of pseudo-Anosov monodromies. Using [22] we can say that the surface bundles  $M_\phi$  are all hyperbolic. By hyperbolic Dehn surgery theorem the surgered manifolds  $M_\phi(r)$  are hyperbolic for all  $m, n \in \mathbb{Z}$  and  $r \in \mathbb{Q}$  except 7 values of  $m$  and finitely many “bad” slopes  $r$ . This finishes the proof of Theorem 1.1.  $\square$

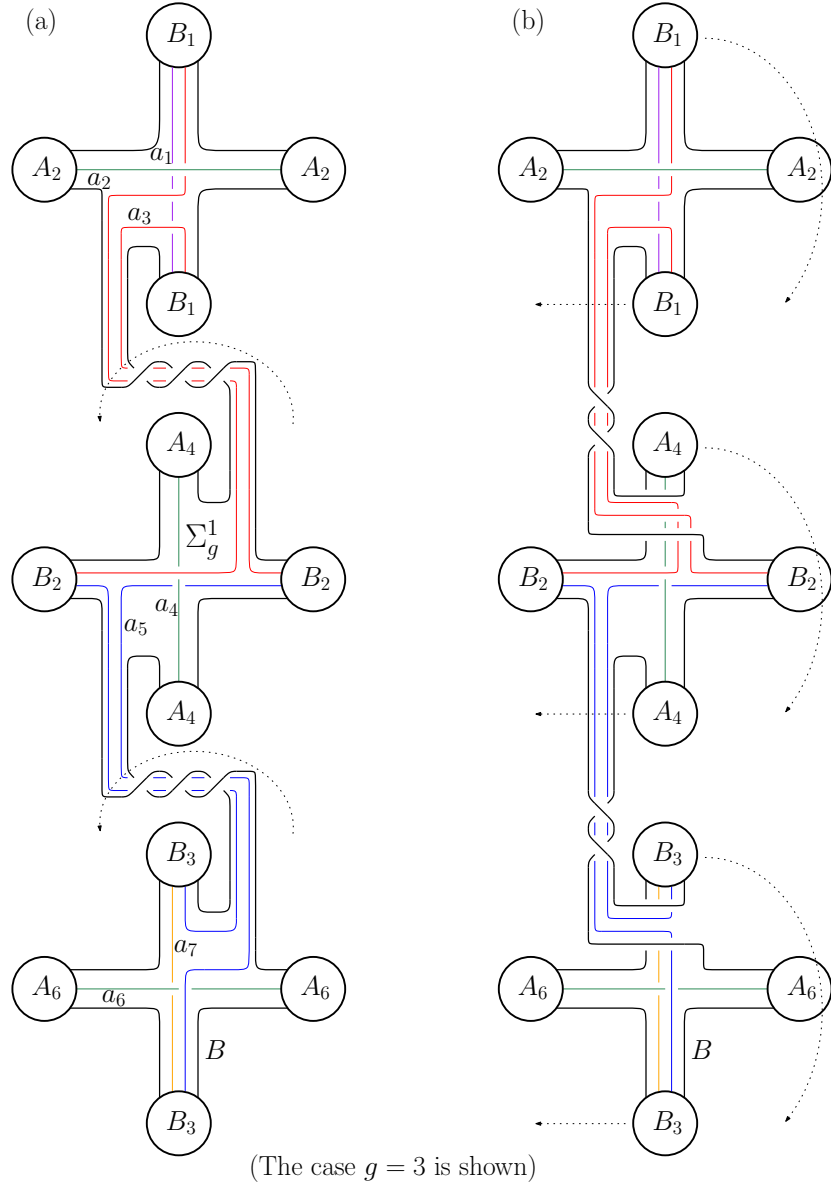


FIGURE 2.

## 3. PROOF OF THEOREM 1.2

We will analyze the proof with respect to the parity of the genus  $g$  of the fiber  $\Sigma_g$ . First assume  $g \geq 3$  odd. Note that conjugation of the monodromy by any class of  $MCG(\Sigma_g)$  does not change the mapping torus up to diffeomorphism. Since

$$t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n t_{a_1}^m = t_{a_1}^{-m} \phi t_{a_1}^m$$

we may replace  $\phi$  in (1) with the mapping class  $t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n t_{a_1}^m$ . Also observe that  $M_\phi(r)$  can be also obtained from a Dehn surgery on the binding of an open book decomposition whose page is  $\Sigma_g^1$  (punctured  $\Sigma_g$ ) and monodromy can be still assumed to be  $\phi \in MCG(\Sigma_g^1)$ . We will construct the required contact structure  $\xi$  on  $M_\phi(r)$  via Dehn surgery on the open book decomposition  $(\Sigma_g^1, \phi)$  along its binding.

It is known (see [1], [14]) that the contact structure, say  $\xi_0$ , (before the surgery along binding) supported by  $(\Sigma_g^1, \phi)$  is Stein fillable. More precisely, consider the handlebody diagram of the smooth 4-manifold  $X_\phi$  given in Figure 2-(a) (in the case of genus 3) with “ $2g$ ” 1-handles and “ $m + n + 2g - 1$ ” 2-handles. Note that Figure 2-(a) describes a Lefschetz fibration structure on  $X_\phi$  with a regular fiber  $\Sigma_g^1$  and the vanishing cycles  $a_1, a_2, \dots, a_{2g+1}$ . There are  $n$  copies for  $a_{2g+1}$  and  $m$  copies for  $a_1$  (not drawn for simplicity). All coefficients (except on  $B$ ) are  $-1$  with respect to the framing given by the page  $\Sigma_g^1$ . We remark that no handle is attached along the binding of the induced open book  $(\Sigma_g^1, \phi)$  on the boundary  $\partial X_\phi$  which is realized as  $B$  in the figure.

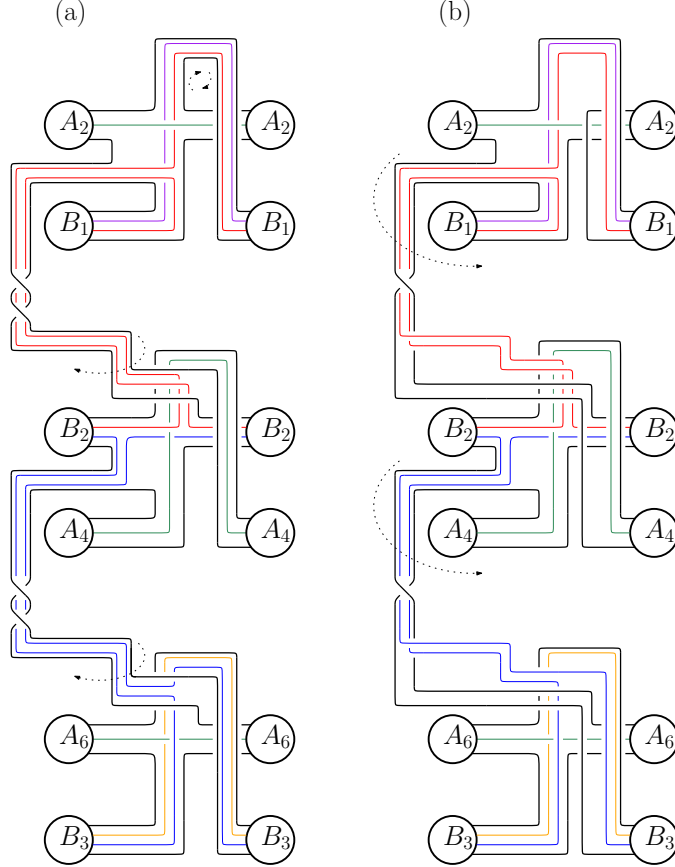


FIGURE 3.

Next starting from the topological description in Figure 2-(a) of  $X_\phi$ , we'll get a diagram describing a Stein structure on  $X_\phi$  inducing  $\xi_0$  as follows: First we flip the twisted bands over the 1-handles as pointed out in Figure 2-(a) and get Figure 2-(b). Figure 3-(a) gives another handle description of  $X_\phi$  obtained by moving the feet of 1-handles as indicated by the dotted arrows in Figure 2-(b). Then flip the bands as shown in Figure 3-(a) to get rid of one more left half twist for each band (see Figure 3-(b)), and obtain Figure 4-(a) by flipping the connecting bands over the feet of 1-handles suggested by the dotted arrows in Figure 3-(b). Figure 4-(b) defines a Stein structure on  $X_\phi$  obtained by putting the attaching circles in part (a) into Legendrian positions, where a Legendrian realization  $L_0$  of  $B$  in the tight contact boundary  $\partial X_\phi$  is also provided. All coefficients (except on  $L_0$ ) are  $-1$  with respect to Thurston-Bennequin (contact) framing in  $\partial X_\phi$  and no handle is attached along  $L_0$ . Note that  $tb(L_0) = 2$  (the case  $g = 3$  is shown). In the general case,  $tb(L_0) = g - 1$ . Finally, we use the trick ("Move 6") in Figure 20 of [15] to obtain a Legendrian representation  $L$  of  $B$  with  $tb(L) = 2g - 1$  (see Figure 5). Note that Figure 5 describes the same Stein structure on  $X_\phi$  as in Figure 4-(b).

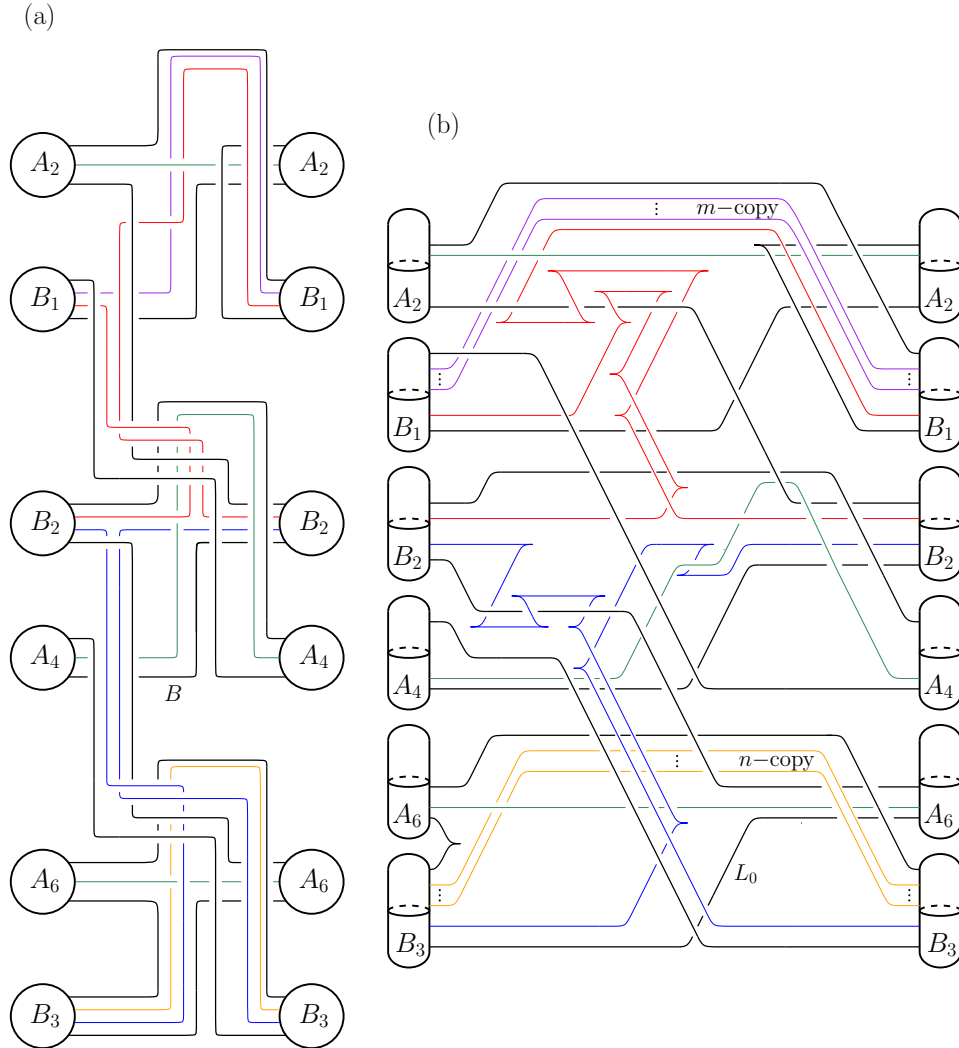


FIGURE 4.

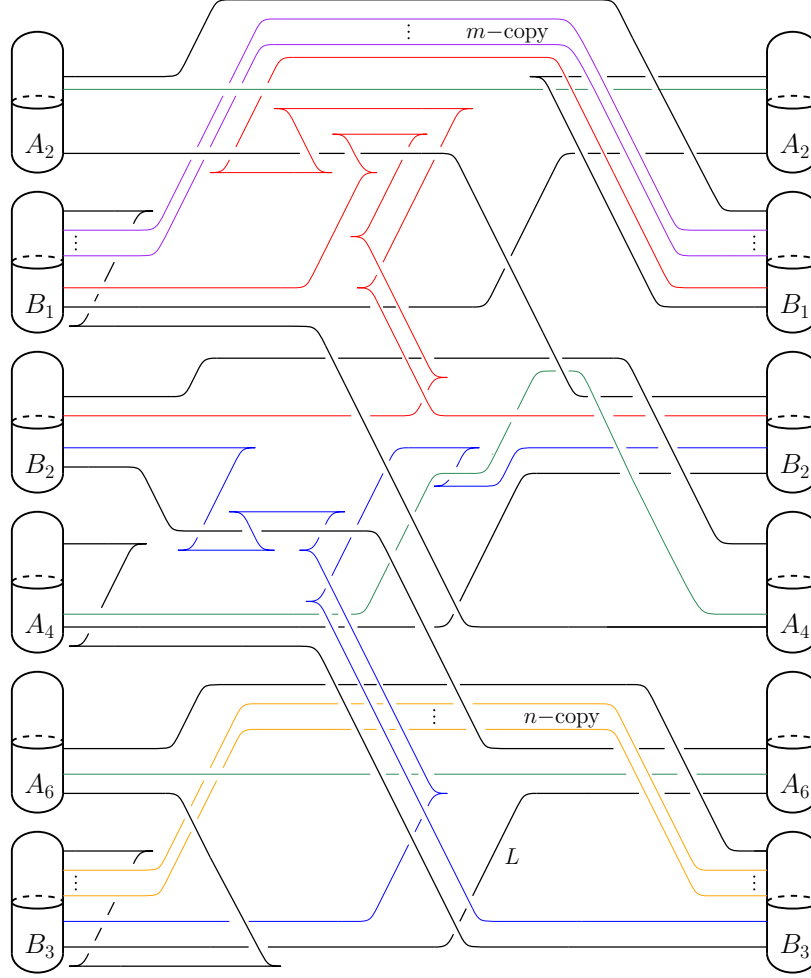


FIGURE 5. The same Stein structure on  $X_\phi$  as in Figure 4-(b), and another Legendrian realization  $L$  of the binding  $B$  in the tight contact boundary  $\partial X_\phi$ .  $L$  is obtained from  $L_0$  by applying “Move 6” (smooth but non-Legendrian isotopy of  $L_0$ )  $g$  times using the left foot of the corresponding 1-handles (when  $g = 3$ , handles are  $B_1, A_4, B_3$ ). All coefficients (except on  $L$ ) are  $-1$  with respect to Thurston-Bennequin (contact) framing in  $\partial X_\phi$ . No handle attached along  $L$ . Note that  $tb(L) = 5$  (the case  $g = 3$  is shown). In the general case,  $tb(L) = 2g - 1$ .

Now if  $g \geq 2$  is even, we replace the monodromy  $\phi$  with  $t_{a_{2g+1}}^n t_{a_2} \cdots t_{a_{2g}} t_{a_1}^m$  since

$$t_{a_{2g+1}}^n t_{a_1}^{-m} \phi t_{a_{2g+1}}^{-n} t_{a_1}^m = t_{a_{2g+1}}^n t_{a_2} \cdots t_{a_{2g}} t_{a_1}^m.$$

Then starting from the handlebody diagram given in Figure 6-(a) (where the case  $g = 4$  is shown) and following the moves as in the case of odd genus, one can get Figure 6-(b) describing a Stein structure realizing a Legendrian representation  $L$  with  $tb(L) = 2g - 1$  as in Figure 5. One should note that we need to consider different monodromies (but still giving the same mapping torus) depending on the parity of  $g$  to make the contact and the page framing on any attaching circle coincide.

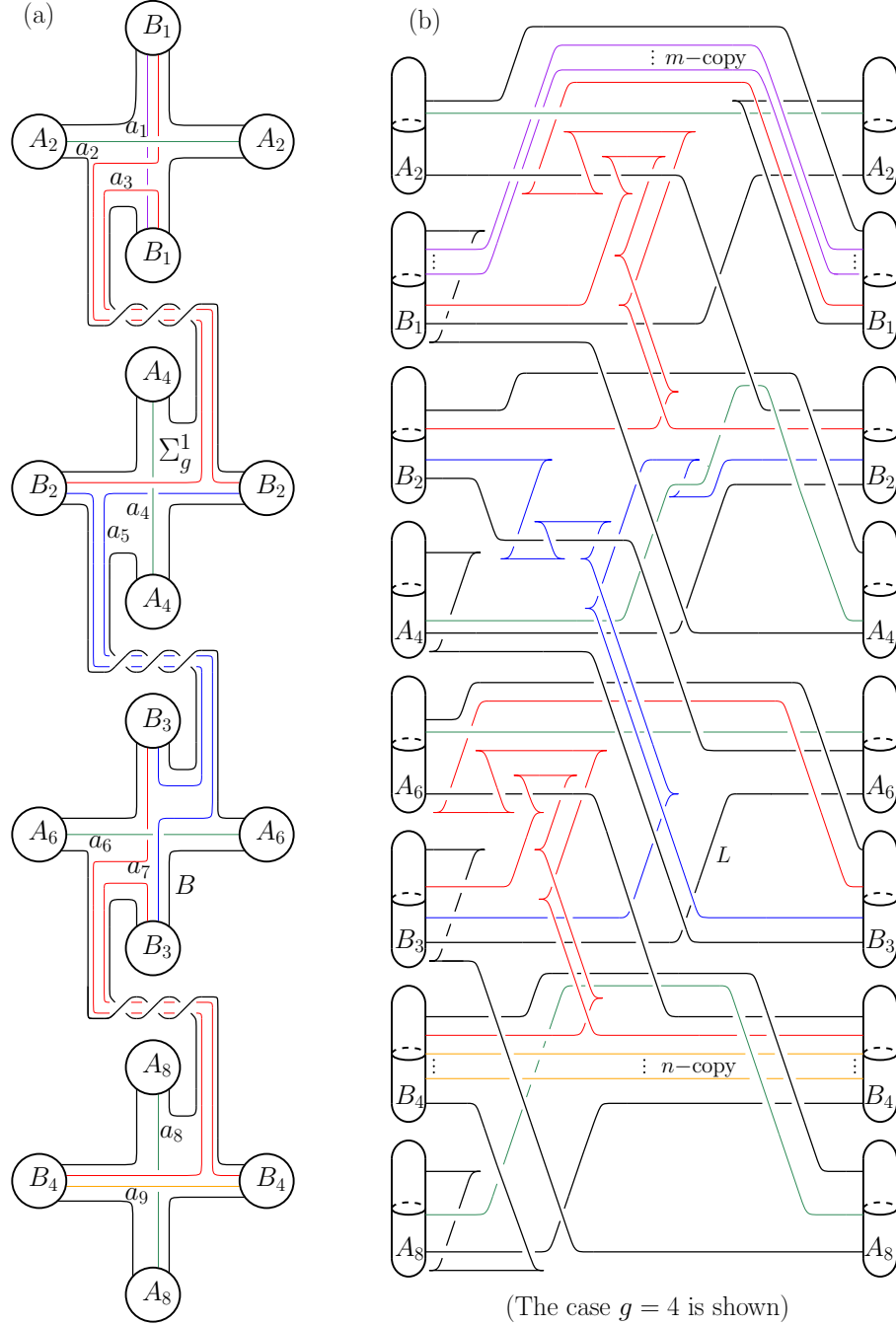


FIGURE 6.

Now (in any case of  $g$ ) we first (Legendrian) slide (Stein) 2–handle corresponding  $a_3$  over the ones represented by the curves  $a_1, a_5, a_7, \dots, a_{2g+1}$ , and then cancel the 2–handles represented by  $a_5, a_7, \dots, a_{2g-1}$  with the corresponding 1–handles. Second, we (Legendrian) slide 2–handles represented by the curves  $a_1$  and  $a_{2g+1}$  over a fixed one (chosen from each family in Figure 5 / Figure 6-(b)), and then cancel 1–handles  $B_1$  and  $B_g$  with the chosen 2–handles corresponding  $a_1$  and  $a_{2g+1}$  respectively. Also we cancel each 1–handle  $A_i$  with the 2–handle corresponding

the curve  $a_i$  for each  $i$  even. As a result, we obtain another (but equivalent) Stein structure on  $X_\phi$  which can be also considered as the contact surgery diagram for  $\xi_0$  on  $\partial X_\phi$ . Finally, we set  $r' = r - 2g + 1$  and perform  $r'$ -contact surgery along  $L \subset (\partial X_\phi, \xi_0)$  to get a contact structure  $\xi$  on  $M_\phi(r)$  whose diagram is given in Figure 7 (where we use continued fractions).

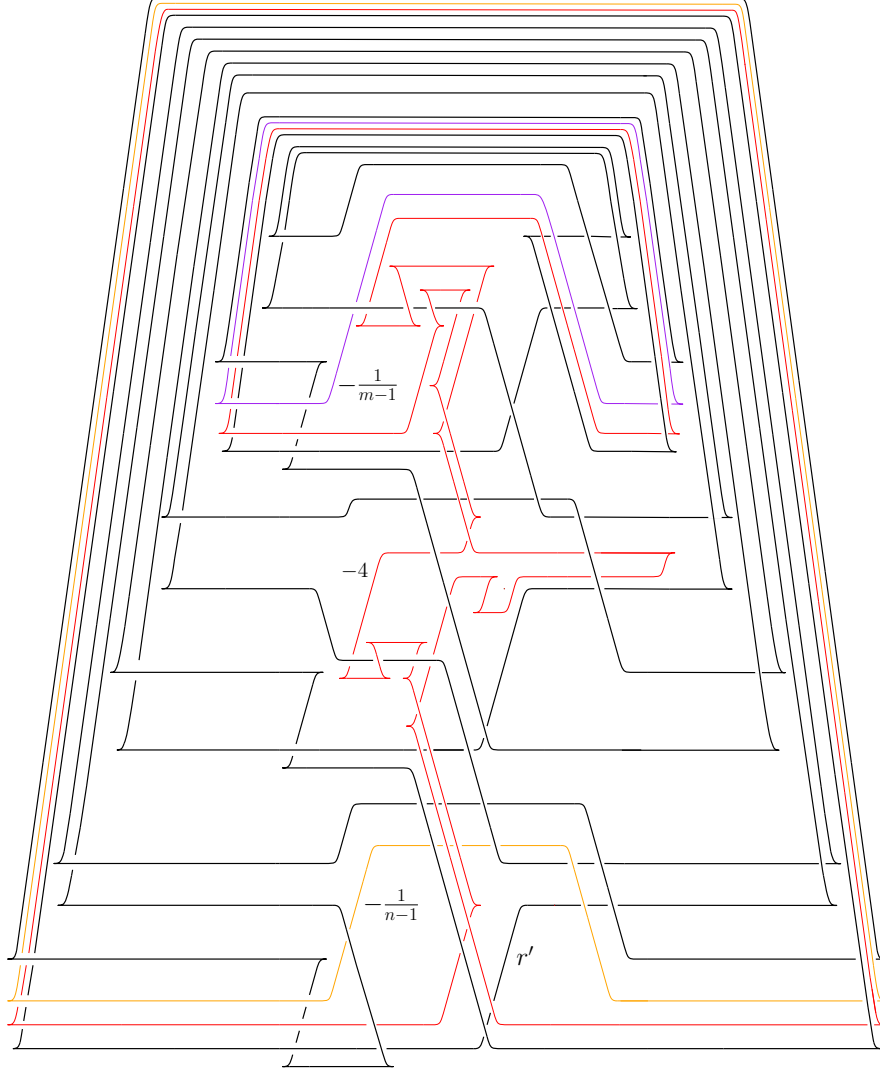


FIGURE 7. The contact 3-manifold  $(M_\phi(r), \xi)$ . (The case  $g = 3$  is shown.)

First suppose  $r' = r - 2g + 1 < 0$ . We know any contact surgery with negative contact framing can be converted to a sequence of contact  $(-1)$ -surgeries and  $(-1)$ -surgeries preserve Stein fillability ([5], [6], [14]). Thus  $(M_\phi(r), \xi)$  is Stein fillable (hence tight).

Now let  $r' = r - 2g + 1 > 0$ . By Thurston-Winkelnkemper construction ([24]), it is known that the binding  $B$  is transverse to the contact structure supported by the open book decomposition. Also since  $\partial X_\phi$  is Stein fillable,  $\xi_0$  has nonzero contact invariant [21]. As a result of Conway's work (see [3], Theorem 1.6) if  $K$  is a fibered transverse knot in a contact 3-manifold  $(M, \eta)$  where  $\eta$  has nonvanishing contact class, then  $r$ -surgery along  $K$  preserves the non-vanishing of



the contact class if  $r > 2g - 1$  where  $g$  is the genus of  $K$ . Hence we conclude that  $(M_\phi(r), \xi)$  has nonzero contact invariant (hence tight) through Conway's result. This finishes the proof of Theorem 1.2.  $\square$

*Acknowledgments.* The authors would like to thank James Conway and Mustafa Korkmaz for their invaluable comments.

## REFERENCES

- [1] S. Akbulut and B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. 5 (2001), 319-334.
- [2] V. Colin, *Une infinite de structures de contact tendues sur les varietes torodales*, Comment. Math. Helv. 76 (2001) 353372.
- [3] J. Conway, *Transverse Surgery on Knots in Contact 3-Manifolds*, arXiv: 1409.7077.
- [4] F. Ding and H. Geiges, *Handle moves in contact surgery diagrams*, J. Topol. 2 (2009), no. 1, 105-122.
- [5] F. Ding, H. Geiges and A. Stipsicz, *Surgery diagrams for contact 3-manifolds*, Turkish J. Math. 28 (2004), no. 1, 4174.
- [6] Y. Eliashberg, *Topological Characterization of Stein manifolds of dimension  $> 2$* , Int. J. Math., 1(1990) 29-46.
- [7] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier 42 (1992) 165192.
- [8] T. Etgu, *Tight contact structures on laminar free hyperbolic three-manifolds*, Int. Math. Res. Not. IMRN 2012, no. 20, 47754784.
- [9] J. B. Etnyre, *Legendrian and transversal knots*, Handbook of knot theory, 105-185, Elsevier B. V., Amsterdam, 2005.
- [10] J. Etnyre, K. Honda, *On the nonexistence of tight contact structures*, Ann. of Math. (2) 153 (2001) 749766.
- [11] J. B. Etnyre and J. Van Horn-Morris, *Fibered transverse knots and the Bennequin bound*, Int. Math. Res. Not. IMRN 2011, no. 7, 14831509.
- [12] A. Fathi, *Dehn twists and pseudo-Anosov diffeomorphisms*, Invent. Math., 87 (1987), no. 1, 129151.
- [13] H. Geiges, *An Introduction to Contact Topology*, Cambridge University Press, (2008).
- [14] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing), Higher Ed. Press, (2002), pp. 405-414. MR 2004c:53144.
- [15] R. E. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. (2) 148 (1998), no. 2, 619693.
- [16] K. Honda, W. Kazez, G. Matic, *Convex decomposition theory*, Int. Math. Res. Not. 2002 (2002) 5588.
- [17] A. Kaloti, B. Tosun, *Hyperbolic rational homology spheres not admitting fillable contact structures*, arXiv:1508.07300.
- [18] P. Lisca and A. Stipsicz, *On the existence of tight contact structures on Seifert fibered 3-manifolds*, Duke Math. J., 148(2):175209, 2009.
- [19] J. Martinet, *Formes de contact sur les varietes de dimension 3*, Proceedings of Liverpool Singularities Symposium, II (1969/1970), pp. 142163. Lecture Notes in Math., 209, Springer, Berlin, 1971.
- [20] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, (1995).
- [21] P. Ozsvath, A. Stipsicz and Z. Szabo, *Planar open books and Floer homology* Int. Math. Res. Not. 2005, no. 54, 33853401.
- [22] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bulletin of the American Mathematical Society, vol. 19 (1988), pp. 417431.
- [23] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton lecture notes (1978-1981).
- [24] W.P. Thurston and H.E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. no. 52 (1975), 345-347.
- [25] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. 20 (1991), no. 2, 241-251.

DEPT. OF MATHEMATICS, MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA, TURKEY  
*E-mail address:* farikan@metu.edu.tr

DEPT. OF MATHEMATICS, ULUDAĞ UNIVERSITY, BURSA, TURKEY  
*E-mail address:* msecgin@uludag.edu.tr